114-E FINAL EXAMINATION

NAME: ________________________________

Prof. Ghrist : Spring 2018

INSTRUCTIONS:

No book or calculators.

Use a writing utensil and logic.

Show all of your work
Explain yourself clearly to receive partial credit.

All problems are equally weighted. Not all take the same amount of time.

Cheating, or the appearance of cheating, will be dealt with severely. By placing your name on this page, you agree to abide by the rules.

Stay calm. All of these problems are doable. You can make it.

Best wishes!

DO NOT

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PROBLEM 1: Consider the following functions:

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y^2 - x^2 \\ xyz \\ x + 2y - z \end{pmatrix} \qquad \& \qquad g\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u + 2v \\ 3w - u \\ w^2 \end{pmatrix}$$

A) Compute $[Df]$ and $[Dg]$:

B) If $u = 3, v = -2, w = 0$, what are the outputs $g(u, v, w)$? [no work needed…]

C) If $u = 3, v = -2, w = 0$, and these three inputs are changing at the rates $\dot{u} = 1, \dot{v} = -2,\ \&\ \dot{w} = 0$, at what rates are the outputs of $f \circ g$ changing? Write your answer as a vector.

ANSWER

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PROBLEM 2: Assume that the expected time-to-completion for a financial transaction is a random variable $0 \leq X < \infty$ with exponential probability density given by: $\rho(x) = e^{-x}$. Assume that there are $n > 1$ such transactions, each with a random variable $0 \leq X_i < \infty$, and these are all independent, so that the joint probability density on the vector $\boldsymbol{X} = (X_1, X_2, \dots, X_n)$ equals

$$\rho(X) = e^{-x_1}e^{-x_2} \dots e^{-x_n} = e^{-x_1-x_2- \dots -x_n}$$

What is the probability that all n transactions complete within 10 time units? (That is, $X_i \leq 10$ for all i). Do this carefully, showing how you set it up as an integral and how you evaluate it. Show your work!

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PROBLEM 3: Consider a particle moving along the following curve in $\mathbb{R}^4$:

$$\gamma(t) = \begin{pmatrix} \cos \pi t \\ 2t(2-t)+1 \\ \sin \pi t \\ t^2 - 2t + 3 \end{pmatrix}$$

A) What is its velocity vector? Compute and simplify.

B) Note that $\gamma(0) = \gamma(2)$. Is the angle between the two velocity vectors (at $t = 0$ and $t = 2$) bigger than or less than or equal to 90 degrees? Explain.

C) Set up but do not solve an integral to determine the arclength of the path traced out by the particle for $0 \leq t \leq 2$. Your answer may be UGLY.

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PROBLEM 4: Consider the following differential form fields:

$$\alpha = 3y\,dx - (x^2 + z^2)\,dz \qquad \& \qquad \beta = 2x\,dy \wedge dz - y\,dx \wedge dy$$

A) Evaluate α at $x = 1, y = -2, z = 3$.

B) Compute and simplify $\alpha \wedge \beta$.

C) Compute and simplify the derivative $d\alpha$.

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PROBLEM 5: Consider the vector field

$$\vec{F} = (2x + x^3)\,\hat{\imath} + (y^3 - 3y)\,\hat{\jmath} + (xy + z^3)\,\hat{k}$$

A) Compute the flux of $\vec{F}$ across the unit sphere centered at the origin. Use an outward-pointing normal to orient the sphere.

ANSWER

B) Why is it that the flux of $\nabla \times \vec{F}$ through this same sphere is zero?

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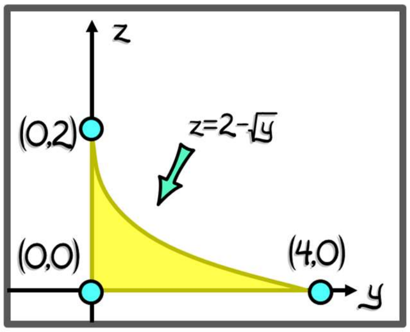
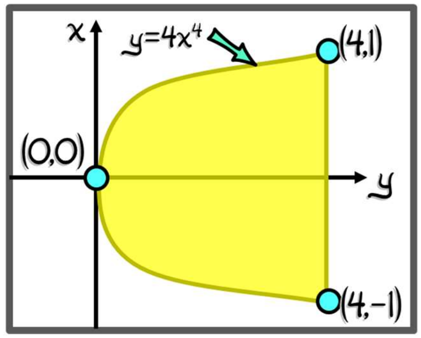
PROBLEM 6: Consider a domain given by the following planar projections:





Write out but do not evaluate an expression involving triple integrals to compute the y-coordinate of the centroid of this region. Use the ordering $dz\ dy\ dx$ for the triple integral and be careful to get the bounds correct!

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PROBLEM 7: If A is the matrix given as…

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & -6 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

Compute the determinant of A^{-1}. Explain/show work!

ANSWER

DO NOT

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PROBLEM 8: Answer the following.

A) For $\boldsymbol{a}$ a critical point of $f(x, y)$, what conditions on the Hessian $[D^2 f]$ guarantee that $\boldsymbol{a}$ is a local maximum?

B) Compute the Taylor series of $\sin xy \cos z$ about the origin $(0,0,0)$, including only terms up to degree [order] ***six***. Show work!

C) Compute $\nabla(\nabla \cdot \vec{F})$ for $\vec{F} = e^x\,\hat{\imath} + \mathrm{y}z\,\hat{\jmath} + (x^2 + y^2)\,\hat{\mathrm{k}}$

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PROBLEM 9: Use Green's theorem to compute the circulation of the planar vector field

$$\vec{F} = (x^2 - y)\,\hat{\imath} + (x - y^2)\,\hat{\jmath}$$

along the closed loop parametrized as $x(t) = 2 + 3\cos t$ & $y(t) = -3\sin t$. Start by drawing a picture of the curve.

ANSWER

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PROBLEM 10: Your goal is to compute an integral over the domain bound by the inequalities:

$$3 \leq x^2 - y^2 \leq 5 \quad \& \quad 1 \leq x^3 y \leq 2 \ .$$

Given these inequalities, it makes sense to use a change of variables to

$$u = x^2 - y^2 \quad \& \quad v = x^3 y \, .$$

A) What is the area element $du\, dv$ in terms of x & y?

ANSWER

B) Set up and carefully solve the integral of $2x^7 y + 6x^5 y^3$ over this domain (defined above) using u & v coordinates. If you did part (A) right, you can do this.

ANSWER

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PROBLEM 11: Consider the following system of linear equations for (u, v):

$$\begin{aligned} 2u + 3v &= 19 \\ 4u + 5v &= 33 \end{aligned}$$

A) Write this in matrix form.

B) Solve the equations using row reduction: show all steps & be neat!

ANSWER

u =

v =

C) Now solve it using the inverse matrix. Since you already know the answer, you had better show your steps carefully…

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PROBLEM 12: Consider the solid object given by the inequalities

$$x^2 + y^2 \leq \cos z \qquad \& \qquad 0 \leq z \leq \frac{\pi}{2}$$

This solid has density $\rho = \sin z$.

A) What is the mass, M, of the solid? (Hint: $x^2 + y^2 = r^2$ in cylindrical…)

ANSWER

B) What is the moment of inertia of this solid about the z-axis?

ANSWER

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PROBLEM 13: Short answers...no justification needed.

A) What are the Lagrange equations for optimizing a function $F(\boldsymbol{x})$ satisfying the constraint $G(\boldsymbol{x}) = 0$?

B) Given an n-by-n matrix, what is the easiest way to tell whether it is invertible?

C) The flux 2-form $\varphi_{\vec{F}}$ associated to the 3-D vector field $\vec{F} = x\,\hat{\imath} + z\,\hat{\jmath} + y\,\hat{k}$ is:

D) What does the scalar triple product of three vectors in $\mathbb{R}^3$ mean geometrically?

E) When integrating a 2-form field β over a parametrized surface $S\begin{pmatrix} s \\ t \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, what pair of vectors does β "eat"?

F) What does the length of the cross product $\boldsymbol{u} \times \boldsymbol{v}$ of two vectors in $\mathbb{R}^3$ tell you?

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PROBLEM 14: Consider the cubic potential function

$$\Phi(x, y, z) = x^2y - 2yz^2 + xyz$$

A) What is the gradient $\nabla\Phi$?

B) Compute the work done by the vector field $\nabla\Phi$ along the parametrized curve:

$$\gamma(t) = \begin{pmatrix} -t \\ t(t+1) \\ t^2 \end{pmatrix} \quad 0 \le t \le 1$$

ANSWER

Show your steps/explain your work.

A PAGE FOR MORE WORK
OR PERHAPS A NICE HAIKU…
WHATEVER YOU WISH